Bayesian estimation of model parameters in a Gaussian plume model using environmental gamma dose rate measurements

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Source inversion

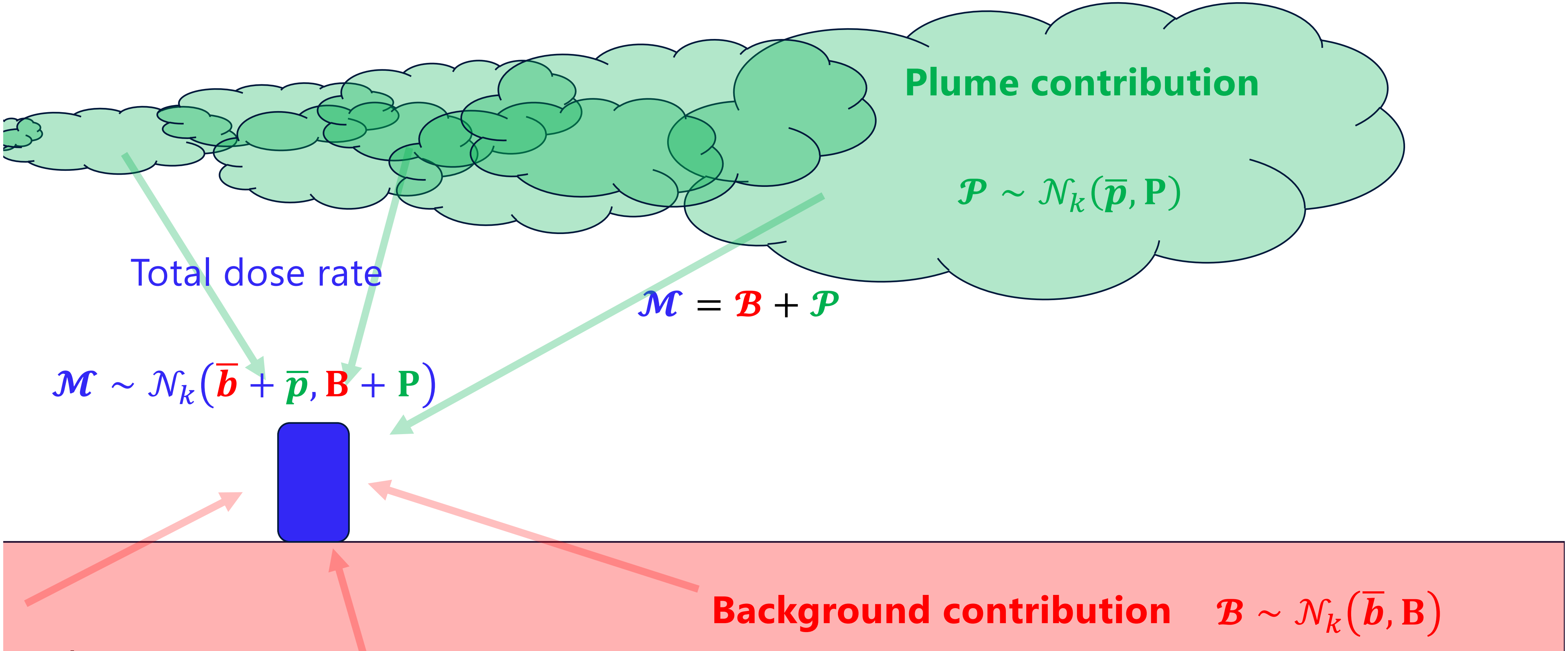
- Source estimation is very much an open topic in the field
- Here, we tackle the question specifically
  - On the near range
  - For gamma dosimetry
- One consistent framework for the background and the plume contribution

Bayesian inference cookbook

$$f_{M|D}(m|d) = \frac{f_{D|M}(d|m)f_M(m)}{f_D(d)}$$

1. Define likelihood $f(d|m)$
2. Define prior $f(m)$
3. Sample posterior $f(d|m)$
4. Predict

Contributions to the total dose rate

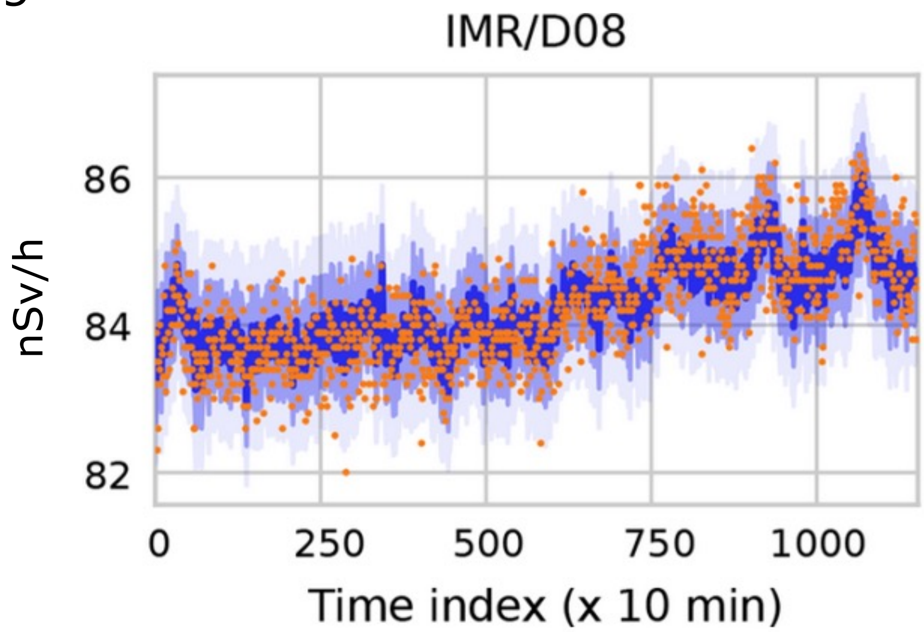


The background model is purely statistical

- Geosci. Model Dev. 18, 1989–2003, 2025
- Parameterise the background as a multivariate normal (likelihood)

$$\boldsymbol{\mathcal{B}} \sim \mathcal{N}_k\left(\overline{\boldsymbol{b}}, \mathbf{B}\right)$$

- Fit arbitrary covariance matrix $\underline{\mathbf{B}}$ using an LKJ Prior
- Posterior sampledusing NUTS (PyMC)



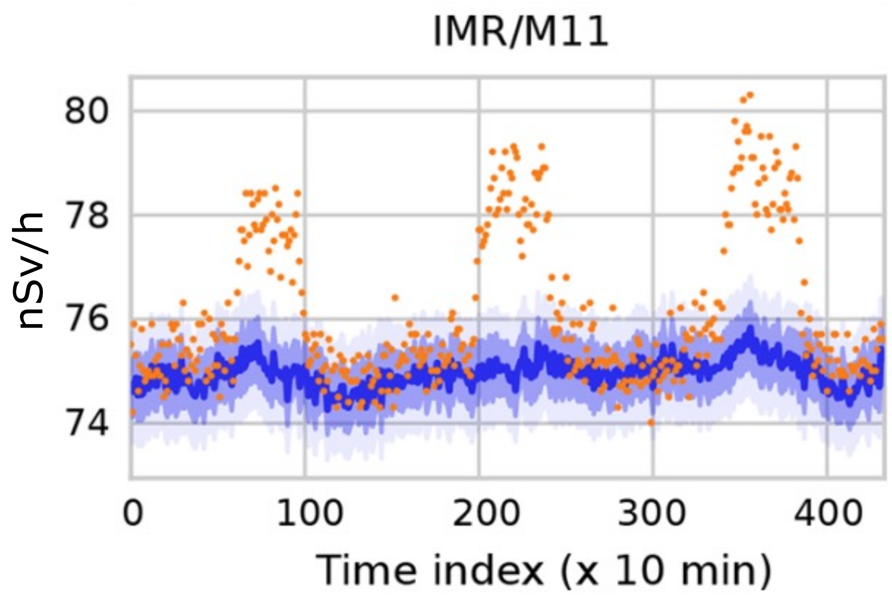
It has a very helpful (analytical!) property

- We can derive a closed-form solution given partial observations ($\boldsymbol{b}_\mathrm{o}$)

$$\mathcal{B}_\mathrm{u} | \boldsymbol{b}_\mathrm{o} \sim \mathcal{N}_{k_\mathrm{u}}\left(\boldsymbol{\mu}_{\mathrm{u|o}}, \mathbf{B}_{\mathrm{u|o}}\right)$$

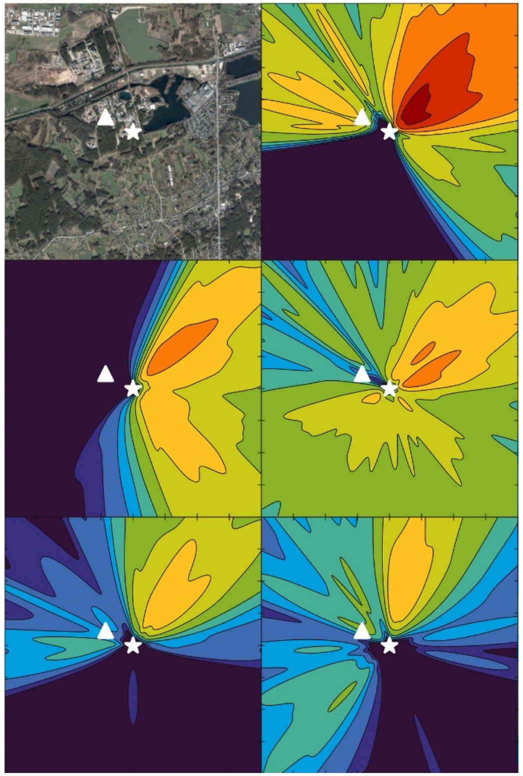
$$\boldsymbol{\mu}_{\mathrm{u|o}} = \boldsymbol{\mu}_\mathrm{u} + \mathbf{B}_{\mathrm{uo}}\mathbf{B}_{\mathrm{oo}}^{-1}(\boldsymbol{b}_\mathrm{o} - \boldsymbol{\mu}_\mathrm{o})$$

$$\mathbf{B}_{\mathrm{u|o}} = \mathbf{B}_{\mathrm{uu}} - \mathbf{B}_{\mathrm{uo}}\mathbf{B}_{\mathrm{oo}}^{-1}\mathbf{B}_{\mathrm{ou}}$$



The plume, however, is not purely statistical

- **A**tmospheric **D**ispersion and **D**ose **E**quivalent **R**ates
- Gaussian plume model with ground and inversion layer reflection
- Customised dispersion parameterisation for the SCK CEN site
- And a high-fidelity finite cloud/ground algorithm



The plume error is modelled as Gaussian as well

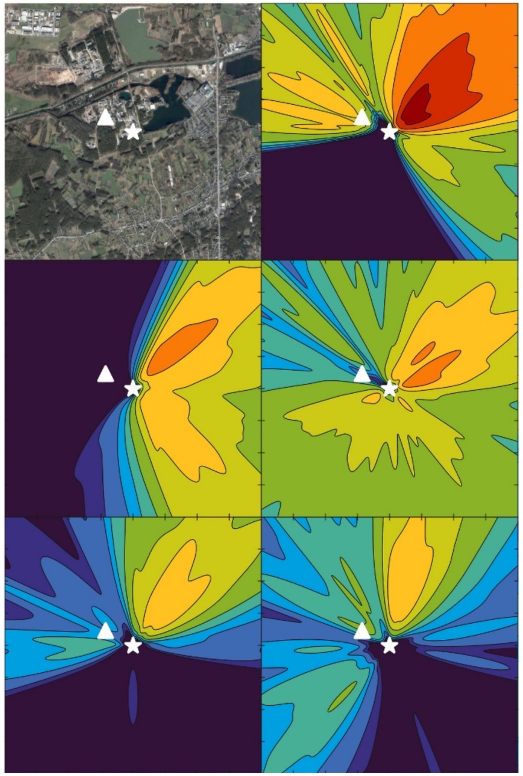
- Likelihood takes the same form as before

$$\mathcal{P} \sim \mathcal{N}_k(\overline{\boldsymbol{p}}, \mathbf{P})$$

- Here, $\overline{\boldsymbol{p}} = \boldsymbol{p}(\boldsymbol{Q}, \dots)$ is the model
- But the covariance matrix is somewhat simpler

$$\mathbf{P} = \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & P_{kk} \end{bmatrix}$$

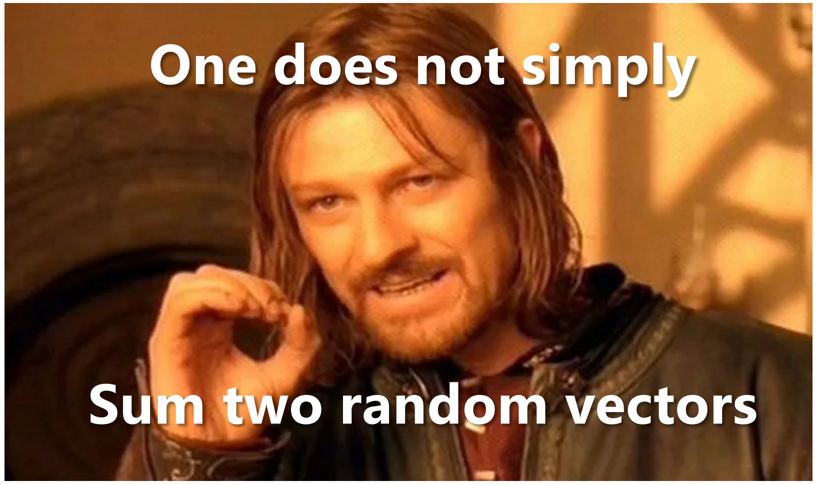
- We set exponential priors (≥ 0) everywhere



Depending on the literature, the model error can be formulated in different ways as well

- Often, the lognormal distribution is preferred over the normal distribution
- However, that comes at a rather large cost in more complex inferences

$$f_{\mathcal{M}}(m) = (f_{\mathcal{B}} * f_{\mathcal{P}})(m) = \int_{\mathbb{R}^k} \mathrm{d}\boldsymbol{\tau}\, f_{\mathcal{B}}(\boldsymbol{\tau})\, f_{\mathcal{P}}(m - \boldsymbol{\tau})$$



Depending on the literature, the model error can be formulated in different ways

- Often, the lognormal distribution is preferred over the normal distribution

- However, that comes at a rather large cost

$$f_{\mathcal{M}}(\boldsymbol{m}) = \frac{1}{(2\pi)^k |\mathbf{B}|^{1/2} |\mathbf{P}|^{1/2} \prod_{i=1}^{k} (\boldsymbol{m}_i - \boldsymbol{\tau}_i)} \int_{\mathbb{R}^k} \mathrm{d}\boldsymbol{\tau} \exp -\frac{1}{2} [g(\boldsymbol{\tau}) + h(\boldsymbol{m} - \boldsymbol{\tau})] \tag{A3}$$

$$g(\boldsymbol{\tau}) = (\boldsymbol{\tau} - \bar{\boldsymbol{b}})^{\top} \mathbf{B}^{-1} (\boldsymbol{\tau} - \bar{\boldsymbol{b}}) \tag{A4}$$

$$h(\boldsymbol{m} - \boldsymbol{\tau}) = (\ln[\boldsymbol{m} - \boldsymbol{\tau}] - \ln \bar{\boldsymbol{p}})^{\top} \mathbf{P}^{-1} (\ln[\boldsymbol{m} - \boldsymbol{\tau}] - \ln \bar{\boldsymbol{p}}) \tag{A5}$$

It is a VERY special property of the (multivariate) normal that this convolution has a closed form

- Exploiting that with gratitude, we thus stick to the formulation

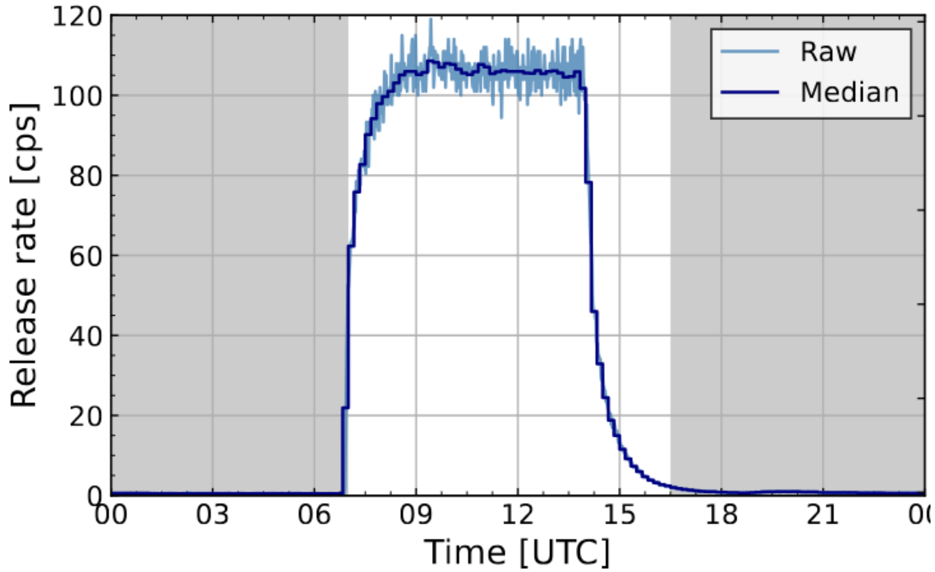
$$\mathcal{M} \sim \mathcal{N}_k(\bar{\boldsymbol{b}} + \bar{\boldsymbol{p}}, \mathbf{B} + \mathbf{P})$$

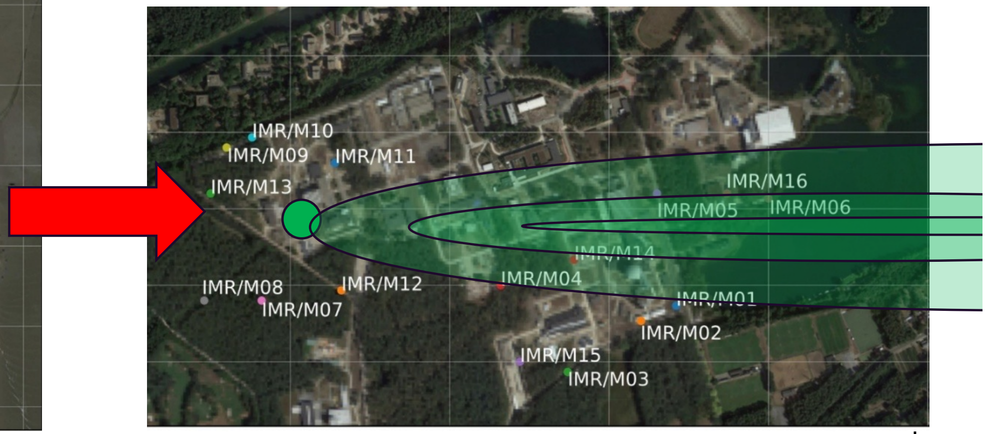
- We choose a two-step approach:
  - First estimate $\bar{\boldsymbol{b}}$ and $\mathbf{B}$ and take the maximum a posteriori values (PyMC)
  - Then estimate $\bar{\boldsymbol{p}} = \bar{\boldsymbol{p}}(\boldsymbol{Q})$ and $\mathbf{P}$ (sampled using TMCMC)

Case study

- 2 weeks of curated of background signal at Doel and Mol
- Day-long argon-41 plume from Belgian Reactor 1
  - 5 hours steady
- On-site meteo info
- In-stack monitor

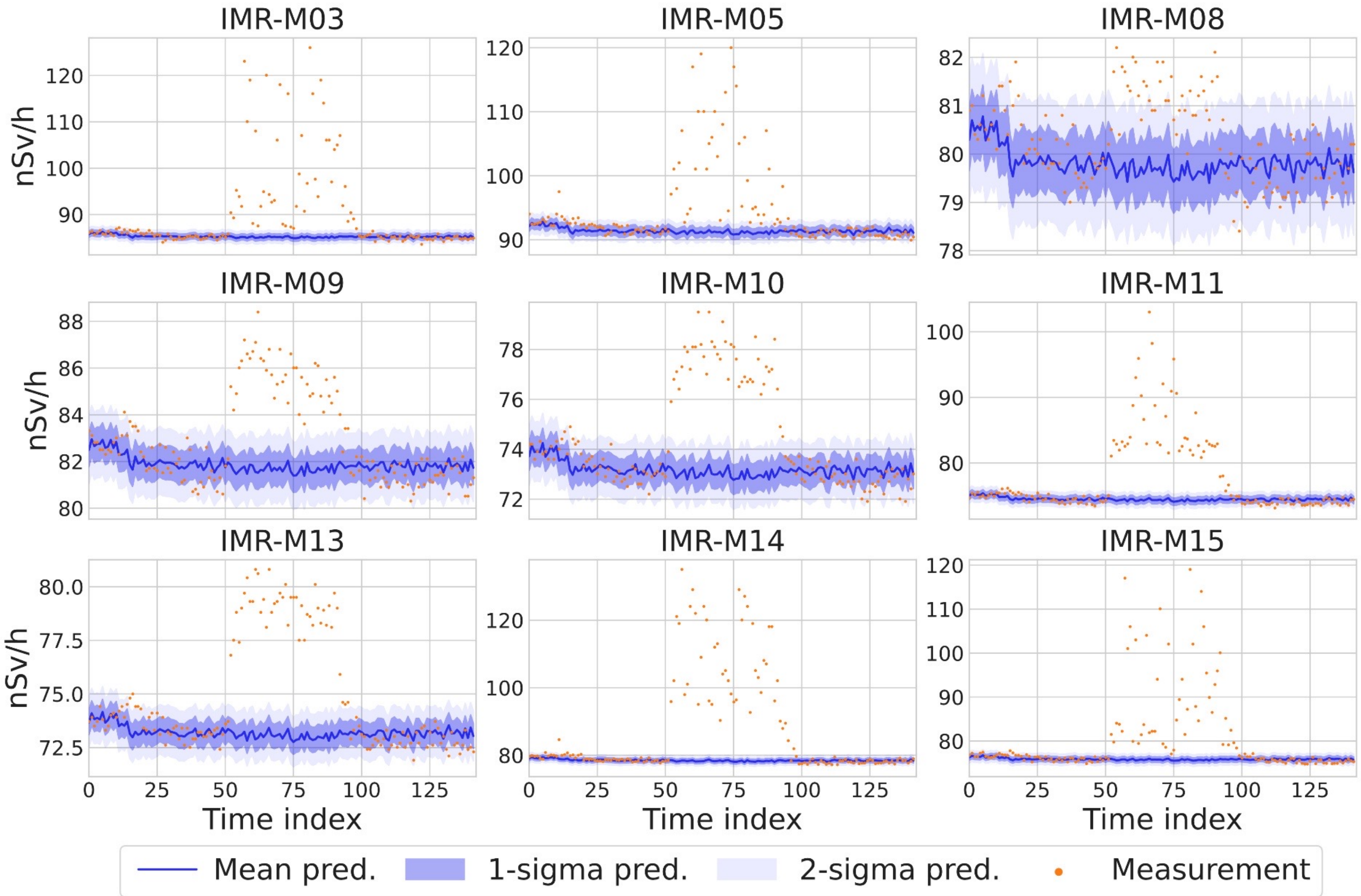
Can we estimate the Source term?





Doel-to-Mol background prediction is excellent

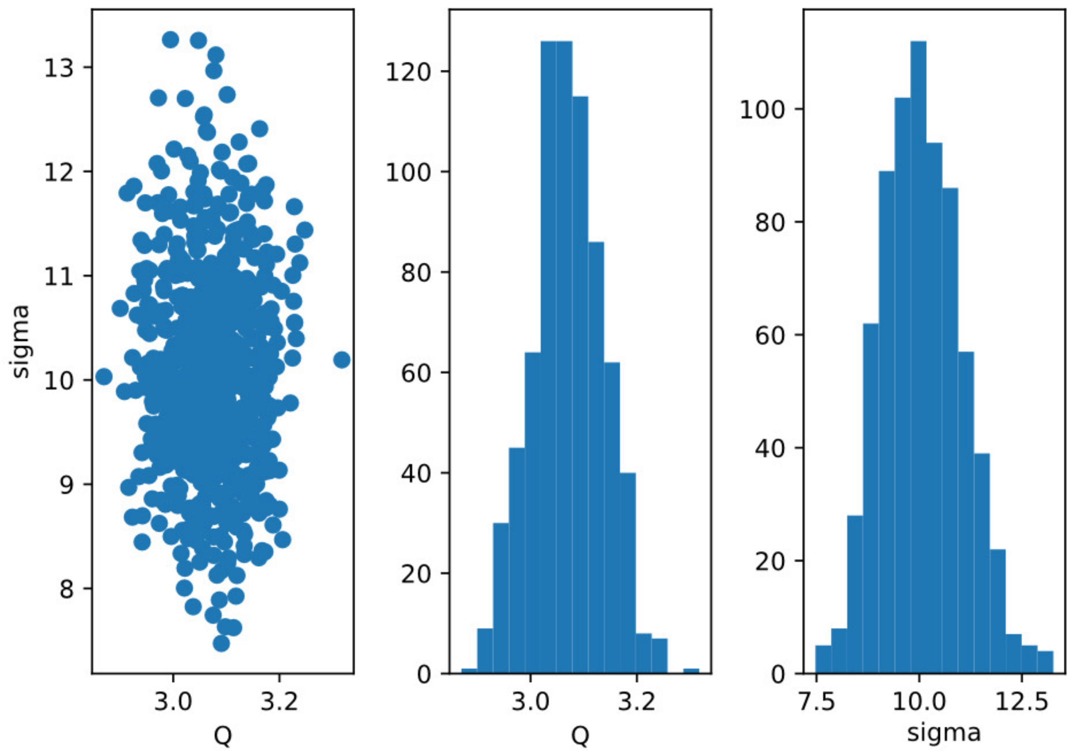
- Happy with estimates for $\bar{b}$ and $\mathbf{B}$
- **They allow isolation of the plume**



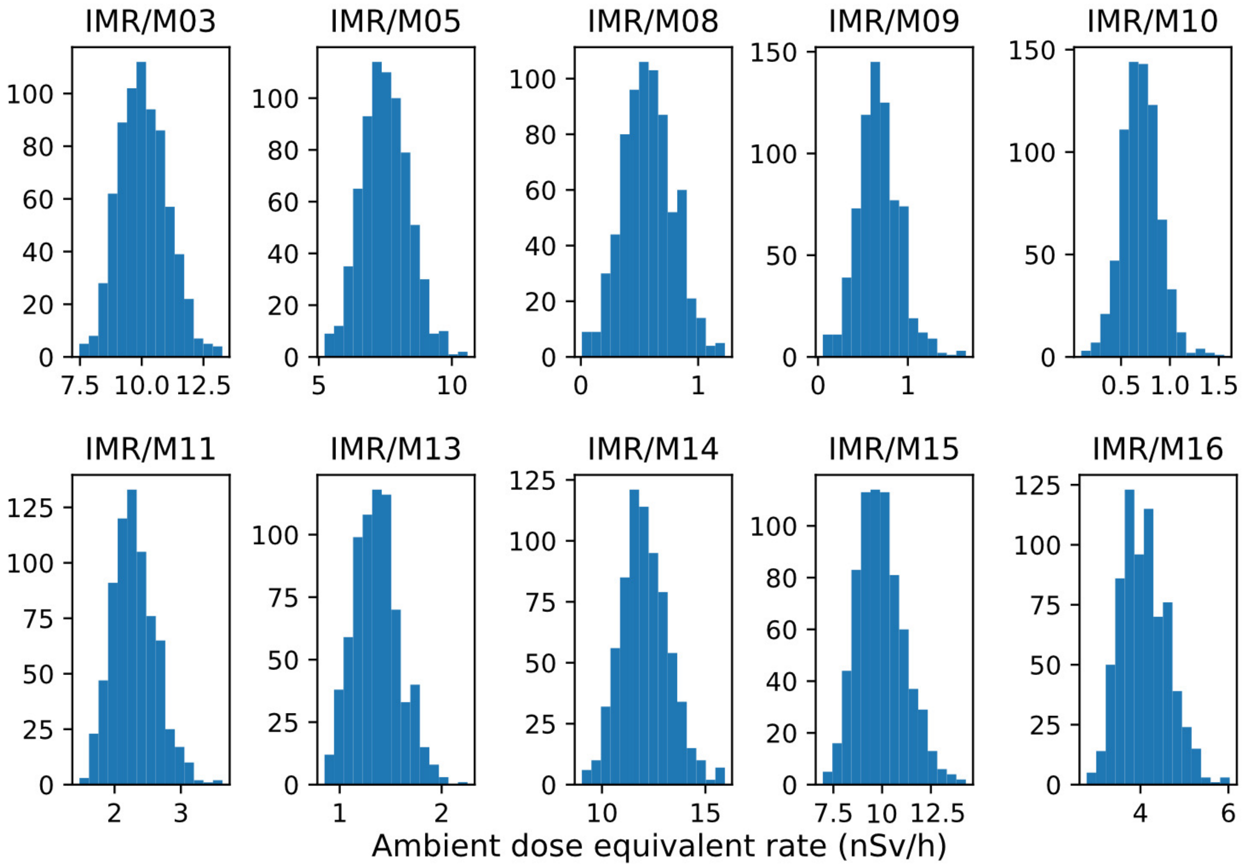
Then, we fix $\overline{b}$ and B, and estimate $\overline{p}(Q)$ and P

- TMCMC sampler
  - 720 samples
  - 144 cores for ~10 min
  - 400 model iterations per core (~100,000 cells)
  - 30 time steps per model iteration (incl. finite cloud)

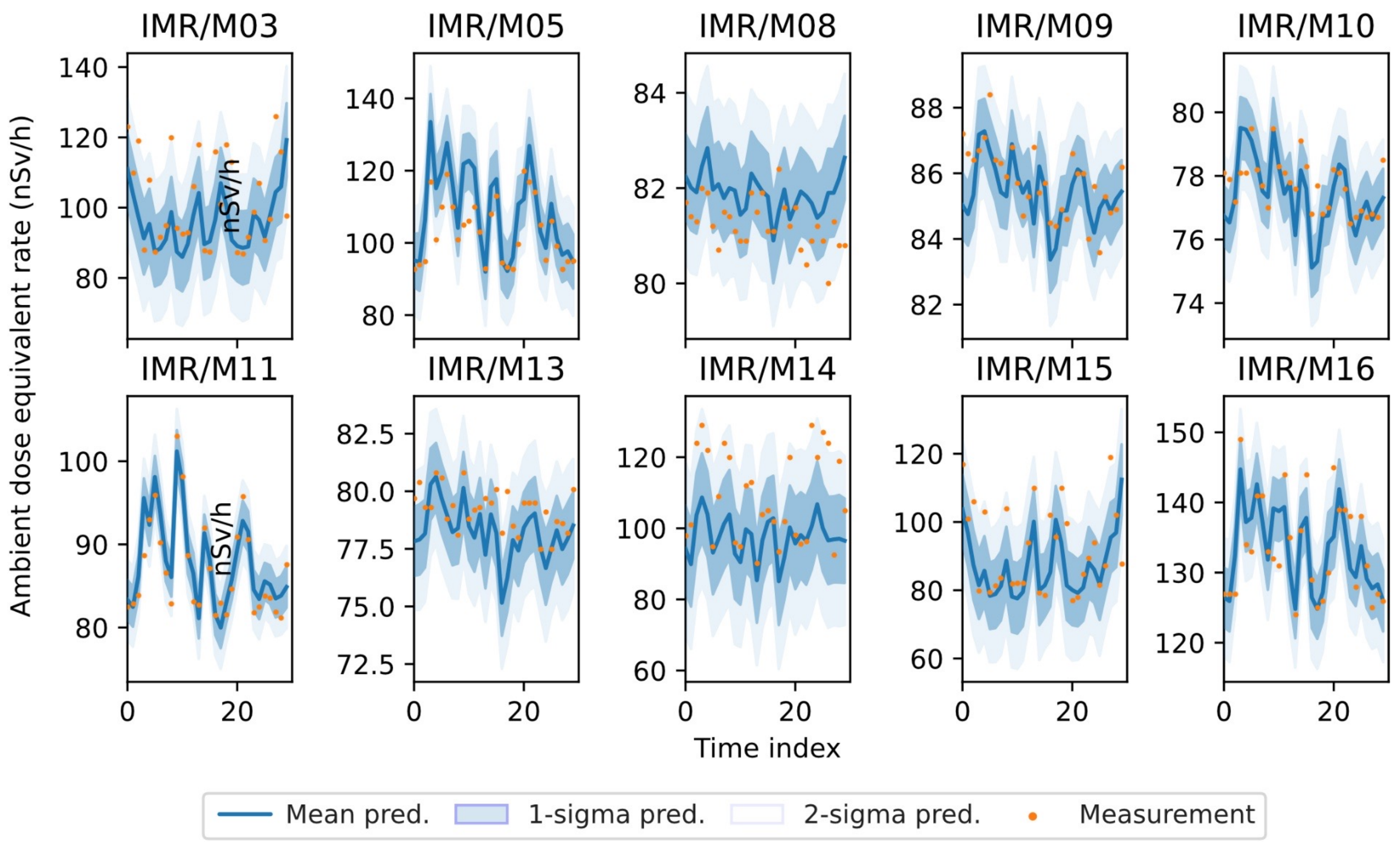
- $Q = release/(57\mathrm{GBq\,h^{-1}})$



Additionally, we get a plume model uncertainty in each of the 10 detectors



Verifying our results via the posterior predictive



Conclusions and outlook

Conclusions

- The background + plume estimation framework functions very nicely
- Fully mathematically consistent framework (hardly any fudging)
- Estimate on the source term including uncertainty margin

Outlook

- Parameterise the plume error in function of space
- That way, forecasting the model error also becomes feasible

Extra slides

Likelihood of the first step

$$\boldsymbol{\mathcal{B}} \sim \mathcal{N}_k\left(\bar{\boldsymbol{b}}, \boldsymbol{B}\right) = \frac{1}{(2\pi)^{k/2}\left|\mathbf{B}\right|^{1/2}} \exp -\frac{1}{2}\left(\boldsymbol{b} - \bar{\boldsymbol{b}}\right)^{\top} \mathbf{B}^{-1}\left(\boldsymbol{b} - \bar{\boldsymbol{b}}\right) = f_{\boldsymbol{\mathcal{B}}}\left(\boldsymbol{b}\right)$$

$$f\left(\bar{\boldsymbol{b}}, \boldsymbol{B} \middle| \boldsymbol{b}\right) = \frac{1}{(2\pi)^{k/2}|\mathbf{B}|^{1/2}} \exp -\frac{1}{2}\left(\boldsymbol{b} - \bar{\boldsymbol{b}}\right)^{\top} (\mathbf{B})^{-1}\left(\boldsymbol{b} - \bar{\boldsymbol{b}}\right)$$

$$f\left(\bar{b}_i\right) = \mathrm{Exp}(100), \mathbf{B} = \mathbf{SRS}, f\left(S_i\right) = \mathrm{HalfNormal}(1)\,, f\left(\mathbf{R}\right) = \mathrm{LKJPrior}(\eta = 1)$$

Likelihood of the second step

$$\boldsymbol{\mathcal{M}} \sim \mathcal{N}_k\left(\bar{\boldsymbol{p}} + \bar{\boldsymbol{b}}, \boldsymbol{B} + \boldsymbol{P}\right)$$
$$= \frac{1}{(2\pi)^{k/2}\,|\mathbf{B}+\mathbf{P}|^{1/2}} \exp -\frac{1}{2}\left(\boldsymbol{m} - \bar{\boldsymbol{b}} - \bar{\boldsymbol{p}}\right)^{\top}(\mathbf{B}+\mathbf{P})^{-1}\left(\boldsymbol{m} - \bar{\boldsymbol{b}} - \bar{\boldsymbol{p}}\right)$$

$$f(Q, \boldsymbol{P}|\boldsymbol{m})$$
$$= \frac{1}{(2\pi)^{k/2}|\mathbf{B}+\mathbf{P}|^{1/2}} \exp -\frac{1}{2}\left(\boldsymbol{m}(Q) - \bar{\boldsymbol{b}} - \bar{\boldsymbol{p}}\right)^{\top}(\mathbf{B}+\mathbf{P})^{-1}\left(\boldsymbol{m}(Q) - \bar{\boldsymbol{b}} - \bar{\boldsymbol{p}}\right)$$
$$f(Q) = \mathrm{Exp}(1), f(P_i) = \mathrm{Exp}(1)$$